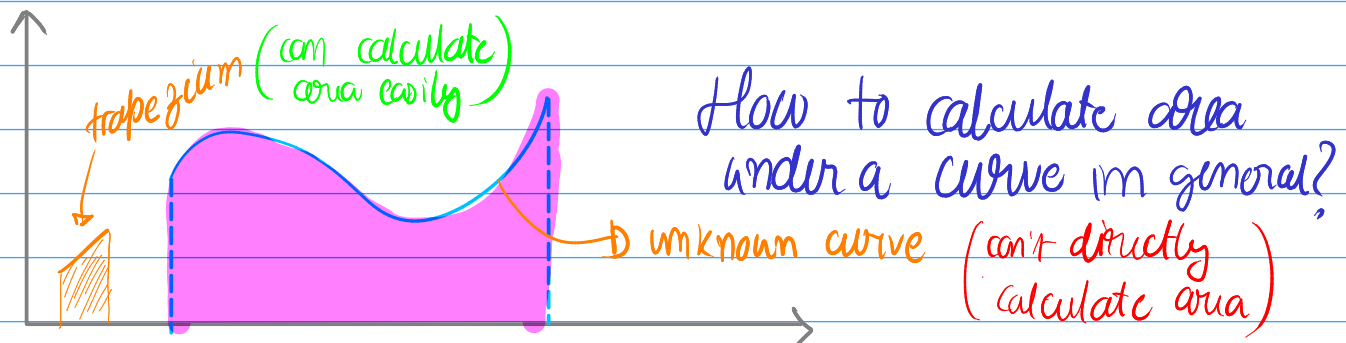


Riemann Integrals

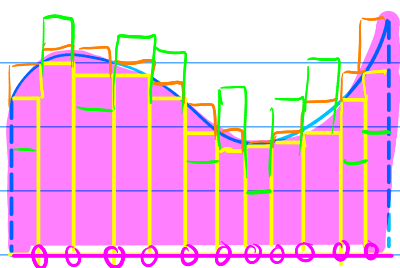
ISI & CMI Yearlong Course 2024

2023-12-27

② Motivation



Importantly, we know how to calculate areas of trapeziums (or more simply, rectangles)



Now, which rectangles to pick?

↳ why equal width rectangles?? Could've done something else.

We can intuitively see that as the number of rectangles increases, our approximation should improve, but there is a lot of freedom in how these rectangles are chosen, and it isn't clear that the approximation works regardless of what choices are made, given some basic assumptions, of course.

We shall spend the next few lectures formalising & refining this idea.

○ Partitions

- The first thing to do is to slice up the interval $I = [a, b]$ into smaller "disjoint" intervals. This is achieved through a partition.
- Defn: A partition of a closed, bounded (viz compact) interval $I = [a, b] \subseteq \mathbb{R}$ is a tuple

$$\mathcal{P} = (x_0, x_1, x_2, \dots, x_{n-1}, x_n)$$

where $n \in \mathbb{N}$, $a = x_0 < x_1 < \dots < x_n = b$; $x_i \in \mathbb{R} \forall i$

The points $\{x_i\}$ are called segmentation points / cut points

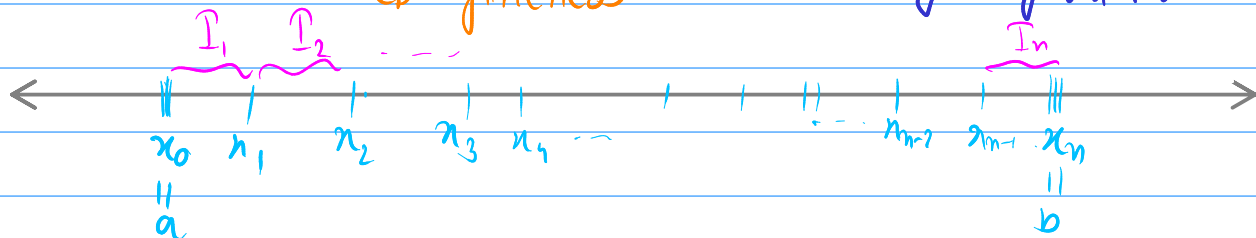
- Defn: $I_j = [x_{j-1}, x_j]$; $j = 1, \dots, n$

segments / sub-intervals.

$$l(I_j) = x_j - x_{j-1} \quad ; \quad j = 1, \dots, n$$

$$\|\mathcal{P}\| = \max_{1 \leq j \leq n} l(I_j) \leftarrow \begin{array}{l} \text{"norm" or "mesh"} \\ \text{of a partition} \end{array}$$

↳ "fineness"



Hence, $I_j \cap I_{j+1} = \{x_j\}$; $\bigcup_{j=1}^n I_j = I = [a, b]$

(now, have to fix heights of each rectangle...)

I - max or min

- take on endpoint

special case of I

- take any value x (bad)

- take value @ any point

we don't use f at all!

◉ Idea I: Take value of f @ any point $\rightarrow$ 'tag'

• Defⁿ: A tagged partition $\dot{P}$ can be defined from $P = (x_0=a, \dots, x_n=b)$ as follows.

$$\dot{P} = \{(\mathcal{I}_j, t_j)\}_{j=1}^n \leftarrow \text{set of 2-tuples.}$$

where $t_j \in \mathcal{I}_j$. $\dot{P}$ $\leftarrow$ signifies that the partition is tagged.

• Defⁿ: $\|\dot{P}\| = \|P\|$ $\hookrightarrow$ ht of rectangles $\sim \mathcal{I}_j$

$$S(f; \dot{P}) = \sum_{j=1}^n \widetilde{f(t_j)} l(\mathcal{I}_j) = \sum_{j=1}^n f(t_j)(x_j - x_{j-1})$$

$\hookrightarrow$ This is called the Riemann sum of $\dot{P}$ wrt f

◉ Idea II: Take the maximum & minimum value of f in segment. (tagging not necessary).

• Defⁿ: $m_j = \inf_{x \in \mathcal{I}_j} f(x)$ $M_j = \sup_{x \in \mathcal{I}_j} f(x)$

• Defⁿ: $L(f; P) = \sum_{j=1}^n m_j l(\mathcal{I}_j) = \sum_{j=1}^n m_j (x_j - x_{j-1})$

$\hookrightarrow$ Lower Darboux summation.

$$U(f; P) = \sum_{j=1}^n M_j l(\mathcal{I}_j) = \sum_{j=1}^n M_j (x_j - x_{j-1})$$

$\hookrightarrow$ Upper Darboux summation.

⊙ Riemann Integrability

- Defn: We say f is Riemann integrable on $[a, b]$ ($f \in \mathcal{R}[a, b]$) iff $\exists L \in \mathbb{R} \mid \forall \varepsilon > 0 \exists \delta > 0$

$$\|\dot{P}\| < \delta \Rightarrow |S(f; \dot{P}) - L| < \varepsilon$$

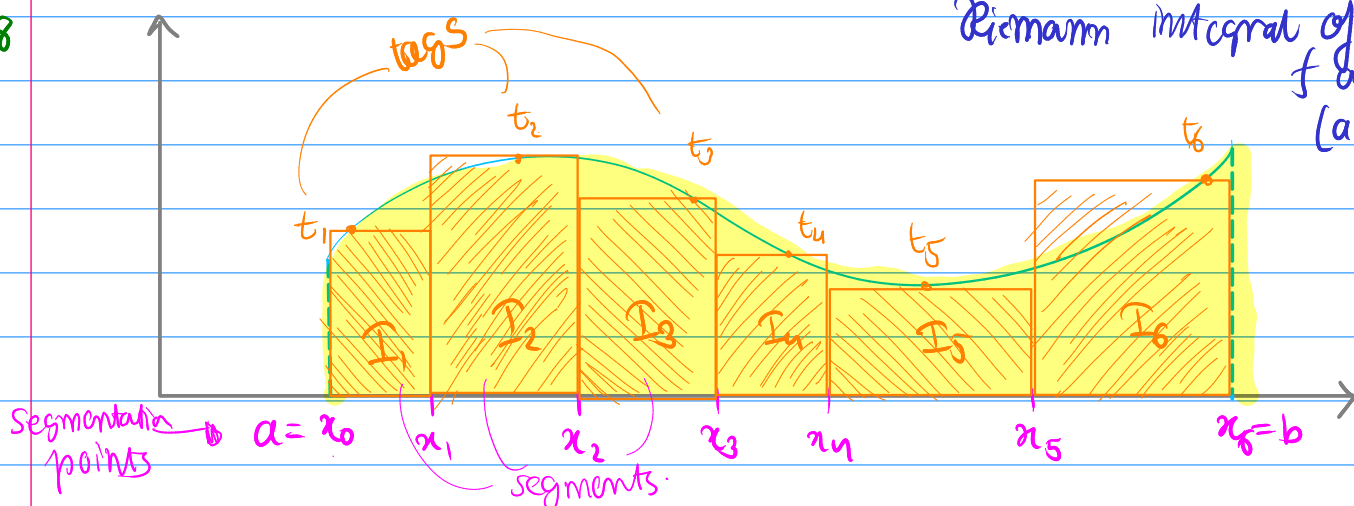
Equivalently, for a sequence of successive refinements $\{\dot{P}_n\}$

$$\|\dot{P}_n\| \rightarrow 0 \Rightarrow S(f; \dot{P}_n) \rightarrow L.$$

(arbitrarily tagged.

& L is called the Riemann integral of f over $[a, b]$

2023-12-28



$$P = (x_0, x_1, x_2, x_3, x_4, x_5, x_6)$$

$$\dot{P} = \left\{ (I_1, t_1), (I_2, t_2), \dots, (I_6, t_6) \right\}$$

$$\parallel$$

$$[x_0, x_1] \qquad \qquad \qquad [x_5, x_6]$$

- NOTE: A partition P^* is said to be a refinement of another P if $P^* \supseteq P$, i.e. every cutpoint of P is also a cut point of P^* . ($P_1 \cup P_2$)
- Observe that i) P^* is a refinement of $P \Rightarrow \|P^*\| \leq \|P\|$
- ii) Any two partitions have a common refinement.

- Theorem: The Riemann integral of f (where $f \in \mathcal{R}[a, b]$) over $[a, b]$ is unique.

→ Proof: Fix $\varepsilon > 0$. Let L, L' be two Riemann integrals.

$\therefore \exists$ tagged partitions $\dot{\mathcal{P}}, \dot{\mathcal{P}}'$ |

$$|S(f; \dot{\mathcal{P}}) - L| < \frac{\varepsilon}{2} \quad \& \quad |S(f; \dot{\mathcal{P}}') - L'| < \frac{\varepsilon}{2}$$

Let $\dot{\mathcal{P}}^*$ be a common refinement of $\dot{\mathcal{P}}$ and $\dot{\mathcal{P}}'$, where subintervals not ^{arbitrarily} containing a tag are tagged with their left endpoint.

Lemma: $\dot{\mathcal{P}}^*$ is a refinement of $\dot{\mathcal{P}} \Rightarrow |S(f; \dot{\mathcal{P}}^*) - L| \leq |S(f; \dot{\mathcal{P}}) - L|$

Using the above Lemma, we have:

$$|S(f; \dot{\mathcal{P}}^*) - L| < \frac{\varepsilon}{2} \quad \& \quad |S(f; \dot{\mathcal{P}}^*) - L'| < \frac{\varepsilon}{2}$$

Applying Triangle Inequality: $|L - L'| < \varepsilon$

$\therefore \varepsilon > 0$ is arbitrary, $L = L'$ ■

{ we could've taken any $\dot{\mathcal{P}}^* \mid \|\dot{\mathcal{P}}^*\| \leq \min(\|\dot{\mathcal{P}}\|, \|\dot{\mathcal{P}}'\|)$ }

→ Since the Riemann integral is unique, it is denoted as $\int_a^b f(x) dx$ where x is called the "variable of integration".
or simply, $\int_a^b f$.

◦ An Alternative Approach: Darboux Summations.

- We have already defined:

$$U(f; \mathcal{P}) = \sum_j M_j \ell(I_j) \text{ where } M_j = \sup_{x \in I_j} f(x)$$

$$L(f; \mathcal{P}) = \sum_j m_j \ell(I_j) \text{ where } m_j = \inf_{x \in I_j} f(x)$$

- We define the upper Riemann integral of f over $[a, b]$ as:

$$\int_a^b f(x) dx = \bar{\int}_a^b f = \inf_{\mathcal{P}} U(f; \mathcal{P})$$

- & define the lower Riemann integral of f over $[a, b]$ as:

$$\int_a^b f(x) dx = \int_a^b f = \sup_{\mathcal{P}} L(f; \mathcal{P})$$

- Defn: We say f is Riemann integrable over $[a, b]$ ($f \in \mathcal{R}[a, b]$) iff:

$$\int_a^b f(x) dx = \bar{\int}_a^b f(x) dx$$

... and the common value is called the Riemann integral of f over $[a, b]$ and denoted as: $\int_a^b f(x) dx$

or simply: $\int_a^b f$.

variable of integration.

- Theorem: The two definitions of Riemann integrability are equivalent.

Proof: left for later.

- Lemma: If $\mathcal{P}'$ refines $\mathcal{P}$, $U(f; \mathcal{P}') \leq U(f; \mathcal{P})$
 $L(f; \mathcal{P}') \geq L(f; \mathcal{P})$

Proof: HW.

- Lemma: For a tagged partition $\dot{\mathcal{P}}$,

$$L(f; \mathcal{P}) \leq S(f; \dot{\mathcal{P}}) \leq U(f; \mathcal{P})$$

Proof: HW (hint: $\forall j \quad m_j \leq f(t_j) \leq M_j$)

- Equivalence Theorem:
 Defn 1 $\leftarrow$ Riemann sum defn
 Defn 2 $\leftarrow$ Darboux sum defn.

$\rightarrow$ Defn 1 $\Rightarrow$ Defn 2 (Refer: 7.4.11 of Bartle-Sharvort)

Given: $\forall \varepsilon > 0 \exists \delta > 0 \mid \forall \dot{\mathcal{P}} \mid \|\dot{\mathcal{P}}\| < \delta, \mid S(f; \dot{\mathcal{P}}) - L \mid < \varepsilon$

TST: $\int_a^b f(x) dx = \int_a^b f(x) dx = L$

Fix $\varepsilon > 0$

We know: $L - \varepsilon < S(f; \dot{\mathcal{P}}) < L + \varepsilon$

$$\& L(f; \mathcal{P}) \leq S(f; \dot{\mathcal{P}}) \Rightarrow L(f; \mathcal{P}) < L + \varepsilon$$

(will move later.

NOTE: In Defⁿ 2, boundedness is implicitly assumed to define m_j, M_j .

Defⁿ 1 $\Rightarrow f$ is bounded on $[a, b]$.

$\therefore \forall j, m_j$ is an infimum, can choose tags t_j

$$f(t_j) < m_j + \frac{\epsilon}{b-a}$$

Noting that $\sum_j l(I_j) = b-a$, we have:

$$\sum_j f(t_j) l(I_j) < \underbrace{\sum_j m_j l(I_j)}_{L(f; \mathcal{P})} + \frac{\epsilon}{(b-a)} \sum_j l(I_j)$$

$$\underline{\text{Now}}, \sum_j f(t_j) l(I_j) = S(f, \mathcal{P}) > L - \epsilon$$

$$\Rightarrow L(f; \mathcal{P}) > S(f, \mathcal{P}) - \epsilon > L - 2\epsilon$$

$$\therefore \text{We have } L - 2\epsilon < L(f; \mathcal{P}) < L + \epsilon$$

As the above holds $\forall \epsilon > 0$, $\sup L(f, \mathcal{P}) = L$

Similarly, $\inf U(f; \mathcal{P}) = L$

Hence, proved.

$\rightarrow$ Defⁿ 2 $\Rightarrow$ Defⁿ 1

skipped for now, might do later.

[Refer to § 7.4 of Bartle-Shurbert for details]

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2023-12-29

- Theorem f is Riemann integrable on $[a, b]$ iff
 $\forall \varepsilon > 0 \exists \mathcal{P} \text{ on } [a, b] \mid U(f; \mathcal{P}) - L(f; \mathcal{P}) < \varepsilon$

→ Proof (If part)

Assume $\forall \varepsilon > 0 \exists \mathcal{P} \mid U(f; \mathcal{P}) - L(f; \mathcal{P}) < \varepsilon$

Take inf over all possible partitions on both sides:

$$0 \leq U(f; \mathcal{P}) - L(f; \mathcal{P}) < \varepsilon$$

$$\Rightarrow 0 \leq \inf [U(f; \mathcal{P}) - L(f; \mathcal{P})] \leq \inf \varepsilon = 0$$

$$\Rightarrow \inf [U(f; \mathcal{P}) - L(f; \mathcal{P})] = 0 \quad \text{--- (1)}$$

Now, $\inf [U(f; \mathcal{P}) - L(f; \mathcal{P})]$

$$\geq (\inf U(f; \mathcal{P})) + (\inf -L(f; \mathcal{P}))$$

$$= \inf U(f; \mathcal{P}) - \sup L(f; \mathcal{P})$$

$$\geq 0$$

But, by (1), $\inf U(f; \mathcal{P}) - \sup L(f; \mathcal{P}) = 0$

$$\Rightarrow \int_a^b f(x) dx = \int_a^b f(x) dx$$

→ Proof (Only 2nd part)

Assume f is Riemann integrable. Fix any $\varepsilon > 0$

$$\therefore \exists \mathcal{P}_1, \mathcal{P}_2 \mid \begin{aligned} U(f; \mathcal{P}_1) &< \int_a^b f(x) dx + \varepsilon/2 \\ L(f; \mathcal{P}_2) &> \int_a^b f(x) dx - \varepsilon/2 \end{aligned}$$

Let $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$ (common refinement)

$$\Rightarrow \begin{aligned} U(f; \mathcal{P}_1) &> U(f; \mathcal{P}) > \int_a^b f(x) dx \\ L(f; \mathcal{P}_2) &< L(f; \mathcal{P}) < \int_a^b f(x) dx \end{aligned}$$

$$\Rightarrow \int_a^b f(x) dx - \frac{\varepsilon}{2} < L(f; \mathcal{P}) < U(f; \mathcal{P}) < \int_a^b f(x) dx + \frac{\varepsilon}{2}$$

$$\Rightarrow U(f; \mathcal{P}) - L(f; \mathcal{P}) < \varepsilon$$

Essentially:

$$\begin{aligned} \int_a^b f - \frac{\varepsilon}{2} &< L(f; \mathcal{P}_2) < L(f; \mathcal{P}) < \int_a^b f \\ \int_a^b f + \frac{\varepsilon}{2} &> U(f; \mathcal{P}_1) > U(f; \mathcal{P}) > \int_a^b f \end{aligned}$$

• Boundedness Theorem ($\Rightarrow$ 1.6 of Bartle-Shurport)

$$f \in \mathcal{R}[a, b] \Rightarrow f \text{ is bounded on } [a, b]$$

Idea is: If f is unbounded on $[a, b]$, for any $\mathcal{P}$,

$\exists j \mid f$ is unbounded on $I_j \Rightarrow$ can choose $t_j \mid f(t_j)$ explodes.

② Algebra of Riemann Integrals

- If $f, g \in \mathcal{R}[a, b]$, $\lambda \in \mathbb{R}$, then $f + \lambda g \in \mathcal{R}[a, b]$
and:

$$\int_a^b (f + \lambda g) = \int_a^b f + \lambda \int_a^b g$$

- If $f, g \in \mathcal{R}[a, b]$, $\forall x \in [a, b]$ $f(x) \leq g(x)$

then:

$$\int_a^b f(x) \leq \int_a^b g(x)$$

- (Squeeze / Sandwich Theorem works, thanks to ↑)

- Additivity Theorem: $f \in \mathcal{R}[a, b] \Leftrightarrow \forall c \in (a, b) f \in \mathcal{R}[a, c] \cap \mathcal{R}[c, b]$

and:
$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$$

where $a < c < b$.

(Proof is too advanced)

- If $f \in \mathcal{R}[a, b]$ then $f \in \mathcal{R}[c, d] \forall [c, d] \subseteq [a, b]$

- Convention: $\int_a^a (\dots) = 0$ & $\int_a^b (\dots) = -\int_b^a (\dots)$

- King's Rule: $\int_a^b f(x) dx = \int_a^b f(a+b-x) dx$

⊙ Classes of Riemann Integrable Functions (§ 7.2 of Bartle-Shrodt)

- Result: Constant functions are Riemann integrable.

Proof: $\forall P, \forall j \quad m_j = M_j \Rightarrow U(f; P) = L(f; P)$

- Result: All step functions (simple fⁿs) are Riemann integrable

... take finite number of discrete values over a bounded set

Hint: First consider elementary step functions:

1 ($x \in \text{interval}$)

Then, show that all step fⁿs are linear combinations of elementary step functions.

$$f(x) = \begin{cases} 1 & \text{in } [1, 2] \\ 2 & \text{in } (2, 3] \\ 3 & \text{in } (3, 4] \\ 0 & \text{o/w} \end{cases} = 1 \cdot \mathbb{1}_{[1, 2]} + 2 \cdot \mathbb{1}_{(2, 3]} + 3 \cdot \mathbb{1}_{(3, 4]}$$

- Result: All continuous functions are integrable. (HW)

- Result: All monotone functions are integrable. (Proof is hard)

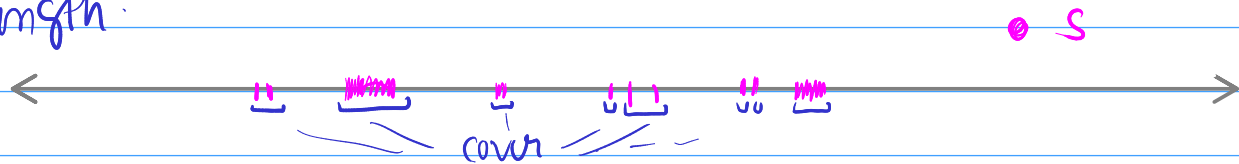
- Lebesgue Integrability Criterion

A function is Riemann integrable iff it is continuous almost everywhere.

↳ i.e. ... except for a measure zero set.

??? TBD.

- Measure zero sets: A set is said to have measure zero iff, it can be covered by a set of intervals of arbitrarily small combined length.



Here, the overall length of the cover is simply the sum of length of all intervals.

Formally, S is covered by $C = \{I_\alpha\} \Leftrightarrow \bigcup_\alpha I_\alpha \supseteq S$

We say S has measure zero when $\forall \varepsilon > 0, \exists C_\varepsilon \mid$

$$\bigcup_{I \in C_\varepsilon} I \supseteq S \quad \text{and} \quad \sum_{I \in C_\varepsilon} l(I) < \varepsilon$$

- ② Prove that $\emptyset$ (the empty set) has measure zero.

Ans: $\forall \varepsilon, C_\varepsilon := \emptyset$ ■

- ② Prove that a finite set has measure zero.

Ans: Let $S = \{x_1, \dots, x_n\}$, fix $\varepsilon > 0$.
 $C_\varepsilon := \left\{ \left[x_1 \pm \frac{\varepsilon}{4n}, x_1 \pm \frac{\varepsilon}{4n} \right], \dots, \left[x_n \pm \frac{\varepsilon}{4n}, x_n \pm \frac{\varepsilon}{4n} \right] \right\}$ works.

- ② Prove that a countable set has measure zero.

Ans: Let $S = \{x_1, x_2, \dots\}$, fix $\varepsilon > 0$
 $C_\varepsilon := \left\{ \left[x_k \pm \frac{\varepsilon}{4} 2^{-k}, x_k \pm \frac{\varepsilon}{4} 2^{-k} \right] \right\}_{k=1}^{\infty}$ works.

→ There are uncountable sets having measure zero too, eg.: Cantor set

- Fact: Altering a f^n on a measure zero set doesn't affect integrability, and doesn't change the value of the integral.

⊙ Defining Integrals through Simple Function Approximations.

- So far, we've seen two approaches to defining integrals: Riemann sums & Darboux sums. However, there is a different approach, which views functions as limiting values of more simpler f^n s. We define integrals for those simpler functions, and use naturally arising properties to extend the defⁿ.

This was inspired by the general defⁿ of expectation as taught by Prof. Mritik Chakraborty, SMU, ISI-K, in his Probability Theory II (Spring 2022) & Probability Theory III (Fall 2022) course, and ideas given by Prof. Krishanu Maulik, SMU, ISI Kolkata, during his Introduction to Stochastic Processes (Spring 2023) course. I am deeply indebted to them for imparting such knowledge to us.

- Idea: ~~Every~~ f^n can be approximated by a sequence of step f^n s (simple f^n s). → not really true, but good enough for our purposes

Define the integral for step f^n s, and extend the defⁿ using $\uparrow$.

- Consider non-negative bounded f's on $[a, b]$.

Define a functional $I(f) = \int_a^b f(x) dx$.

We seek to establish the domain of I and ensure it is well defined on it.

- Step 1: $\forall [c, d] \subseteq [a, b]$

$$I(\mathbb{1}_{[c,d]}) = \dots = I(\mathbb{1}_{(c,d)}) = d - c.$$

- Step 2: Assume I to be a linear functional.

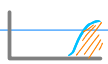
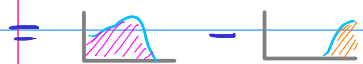
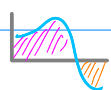
This extends I to all step functions.

- Step 3: For any non-negative bounded f , $\exists$ step f's $\{\psi_k\}_{k \in \mathbb{N}}$, $\psi_k \rightarrow f$ pointwise.

Define $I(f) := \lim_{k \rightarrow \infty} I(\psi_k)$. ← assume measurable.
← whenever this limit makes sense (i.e. integrable f)

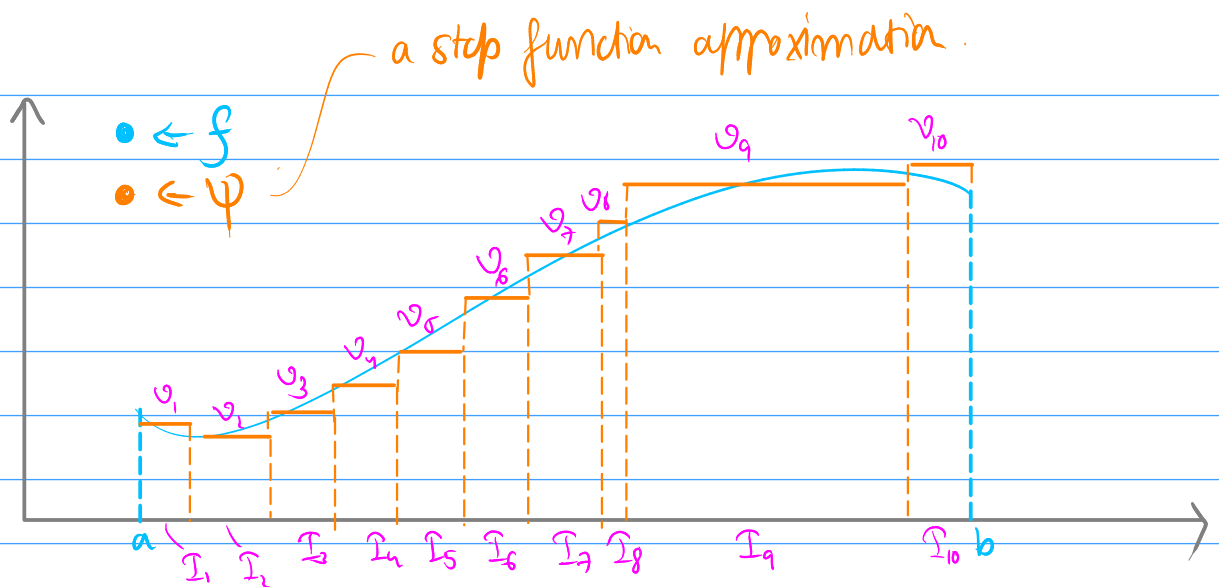
- Step 4: For any bounded f , we write:

$$f(x) = \underbrace{f(x) \mathbb{1}(f(x) \geq 0)}_{f_+(x) \geq 0} - \underbrace{[-f(x)] \mathbb{1}(f(x) < 0)}_{f_-(x) \geq 0}$$



By linearity of I : $I(f) = I(f_+) - I(f_-)$

$$= \int_a^b f(x) dx$$



We know how to integrate Ψ :

$$\int_a^b \Psi(x) dx = \sum_j v_j \ell(I_j) \quad \text{where } \Psi: I_j \mapsto v_j$$

Does this seem familiar?

Remember, Ψ approximates f , so maybe $v_j = f(t_j)$

$$\begin{aligned} \text{Then: } \mathcal{I}(\Psi) &= \sum_j f(t_j) \ell(I_j) \quad \text{where } t_j \in I_j \\ &= S(f; \mathcal{P}) \quad \text{where } \{\Psi\} \approx \{\mathcal{P}\} \end{aligned}$$

Intuitively, $\|\mathcal{P}\| \rightarrow 0$ as $\Psi \rightarrow f$.

Also, $\int_a^b \Psi(x) dx \rightarrow \int_a^b f(x) dx$ as $\Psi \rightarrow f$.

↳ Conventionally Tonelli's Theorem, some conditions are attached (that's why non-negative f).

Hence, this is analogous to Riemann sums!

Convergence of Integrals (later)

✓ MCT ✓ DCT ✓ BCT ✓ Fatou etc.